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The quasi-Zariski topology-graph on the maximal spectrum of modules over commutative rings

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Abstract

Let M be a module over a commutative ring and let $Max(M)$ be the collection of all maximal submodules of M. We topologize $Max(M)$ with quasi-Zariski topology, where M is a Max-top module. For a subset T of $Max(M)$, we introduce a new graph $G(\tau_T^{*m})$, called the quasi-Zariski topology-graph on the maximal spectrum of M. It helps us to study algebraic (resp. topological) properties of M (resp. $Max(M)$) by using the graphs theoretical tools.

1 Introduction

Throughout this paper R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) M. Also $\Lambda(M)$ denotes the set of all non-zero submodules of M. For any pair of submodules $N \subseteq L$ of M and any element m of M, we denote L/N and the residue class of m modulo N in M/N by $\bar{L}$ and $\bar{m}$, respectively.
For a submodule N of M, the *colon ideal of M into N* is defined by $(N : M) = \{r \in R | rM \subseteq N\} = Ann(M/N)$. Further if I is an ideal of R, the submodule

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$(N :_M I)$ is defined by $\{m \in M : Im \subseteq N\}$. Moreover, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the set of positive integers, the ring of integers, and the field of rational numbers.

A prime submodule of M is a submodule $P \neq M$ such that, whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [25].

The prime spectrum (or simply, the spectrum) of M is the set of all prime submodules of M and denoted by $Spec(M)$. Also, the set of all maximal submodules of M is denoted by $Max(M)$.

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $\sqrt{N}$ is defined to be M. Note that the intersection of all prime submodule M is denoted by $rad(M)$.

For a proper ideal I of R, we recall that the J-radical I, denoted by $J^m(I)$, is the intersection of all maximal ideals containing I. The J-radical of a submodule N of M, denoted by $J^m(N)$, is the intersection of all members of $V^m(N)$. In case that $V^m(N) = \emptyset$; we define $J^m(N) = M$.

The *quasi-Zariski topology* on $X := Spec(M)$ is described as follows: put $V^*(N) = \{P \in X : P \supseteq N\}$ and $\xi(M) = \{V^*(N) : N \text{ is a submodule of } M\}$. Then there exists a topology τ^* on X having ξ^* as the set of closed subsets of $Spec(M)$ if and only if ξ^* is closed under the finite union. When this is the case, τ^*_M is called the *quasi-Zariski topology* on $Spec(M)$ and M is called a top module [26].

The *quasi-Zariski topology* on $Max(M)$ does not always exist for any R-module and if it exists, it is called by Max-top module and this topology having $Z^{*m}(M) = \{V^{*m}(N) : N \leq M\}$ as the set of closed sets of $Max(M)$, where $V^{*m}(N) = \{Q \in Max(M) : Q \supseteq N\}$. We denote this topology by τ^{*m}_M. In fact τ^{*m}_M is the same as the subspace topology induced by τ^*_M on $Max(M)$ [22].

If $Max(M) \neq \emptyset$, the mapping $\psi : Max(M) \to Max(R/Ann(M))$ such that $\psi(Q) = (Q : M)/Ann(M) = \overline{(Q : M)}$ for every $Q \in Max(M)$, is called the *natural map* of $Max(M)$ [14].

A topological space X is said to be connected if there doesn't exist a pair U, V of disjoint non-empty open sets of X whose union is X. A topological space X is irreducible if for any decomposition $X = X_1 \cup X_2$ with closed subsets X_i of X with $i = 1, 2$, we have $X = X_1$ or $X = X_2$. A subset X' of X is connected (resp. irreducible) if it is connected (resp. irreducible) as a subspace of X.

Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between ring-theoretic and graph-theoretic properties; see the recent survey articles [1, 24]. The zero-divisor graph of R, $\Gamma(R)$, is a graph with the vertex set $Z(R) \setminus \{0\}$, the set of nonzero

zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of the zero-divisor graph was first introduced by Beck (see [17]), who let all the elements of R be vertices and was mainly interested in colorings. However, the emphasis on the interplay between ring-theoretic properties of R and graph-theoretic properties of $\Gamma(R)$ are from Anderson and Livingston (1999) (see [6]). For a recent article on a related graph to the zero-divisor graph (see annihilator graph as in [15]). Since most properties of a ring are closely tied to the behavior of its ideals, it is worthy to replace the vertices of the zero-divisor graph by the non-zero annihilators ideals. The idea of a graph that whose vertices are a subset of ideals of a ring, was introduced recently in [20]. They defined $AG(R)$, the annihilating-ideal graph of R, to be a graph whose vertices are ideals of R with non-zero annihilators and in which two vertices I and J are adjacent if and only if $IJ = 0$. For other related graphs, we recommend [2, 3, 4, 5, 16].

Let N and L be submodules of M. Then the product of N and L is defined by $(N : M)(L : M)M$ and denoted by NL. Clearly $N^k = (N : M)^k M$ (see [8]).

In [9, 10], the present authors generalized the above idea and introduced the annihilating-submodule graph $AG(M)$ and investigated some of its related properties. The (undirected) graph $AG(M)$ is a graph with vertices $V(AG(M))= \{N \leq M :$ there exists a non-zero proper submodule L of M with $NL = 0$ $\}$, where distinct vertices N, L are adjacent if and only if $NL = 0$.

As we know, the closed subset $V^{*m}(N)$, where N is a submodule of M, plays an important role in the quasi-Zariski topology on $Max(M)$. Our main purpose in this article is to employ these sets and define a new graph $G(\tau_T^{*m})$, called the *quasi-Zariski topology-graph on the maximal spectrum of M*. By using this graph, we study algebraic (resp. topological) properties of M (resp. $Max(M)$). Further we investigate the relationship between $G(\tau_T^{*m})$ and $AG(M/\Im(T))$, where T denotes a non-empty subset of $Max(M)$ and $\Im(T)$ is the intersection of all members of T.

$G(\tau_T^{*m}$) is an undirected graph with vertices $V(G(\tau_T^{*m}))= \{N < M :$ there exists $K < M$ such that $V^{*m}(N) \cup V^{*m}(K) = T$ and $V^{*m}(N), V^{*m}(K) \neq T\}$, where T is a non-empty subset of $Max(M)$ and distinct vertices N and L are adjacent if and only if $V^{*m}(N) \cup V^{*m}(L) = T$ (see Definition 2.1).

Let M be a Max-top module. In section two of this article, among other results, it is shown that the quasi-Zariski topology-graph $G(\tau_T^{*m})$ is connected and $diam(G(\tau_T^{*m})) \leq 3$. Further if $G(\tau_T^{*m})$ contains a cycle, then $gr(G(\tau_T^{*m})) \leq 4$ (see Theorem 2.8). Also, it is shown that $G(\tau_T^{*m})$ has a bipartite subgraph (see Theorem 2.15).

The section three, reflects some fundamental properties of the annihilating-submodule graph of a module which will be used in this paper.

In section four, the relationship between $G(\tau_T^{*m})$ and $AG(M/\Im(T))$ is investigated. We show that if N and L are non-zero proper submodules of M which are adjacent in $G(\tau_T^*)$, then $J^m(N)/\Im(T)$ and $J^m(L)/\Im(T)$ are adjacent in $AG(M/\Im(T))$ (see Proposition 4.5). Further it is proved that if M is a fully semiprime module, then $G(\tau_T^{*m})$ is isomorphic with a subgraph of $AG(M/\Im(T))$ (see Theorem 4.6).

Let us introduce some graphical notions and denotations that are used in what follows: A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non-empty set of vertices $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. The degree $d_G(x)$ of a vertex x is the number of edges incident with x. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i. The number of edges crossed to get from x to y in a path is called the length of the path. A graph G is connected if a path exists between any two distinct vertices. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such path $d(x, y) = \infty$. The diameter of G is $diam(G) = sup\{d(x, y) : x, y \in V(G)\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycle)(see [6]).

A graph H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to $E(H)$. We denote the complete graph on n vertices by K_n. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U (see [28]).

In the rest of this article, M denotes a Max-top module, T a non-empty subset of $Max(M)$, $\Im(T)$ is the intersection of all members of T, $\hat{M}$ represents the R-module $M/\Im(T)$, and for a submodule N of M, $\hat{N} = N/\Im(T)$, where $\Im(T) \subseteq N$.

2 The qausi-Zariski topology-graph on the maximal spectrum of a module

Definition 2.1. We define $G(\tau_T^{*m})$, the *quasi-Zariski topology-graph on the maximal spectrum of* M with vertices $V(G(\tau_T^{*m}))$= $\{N < M$: there exists $K < M$ such that $V^{*m}(N) \cup V^{*m}(K) = T$ and $V^{*m}(N), V^{*m}(K) \neq T\}$, where distinct vertices N and L are adjacent if and only if $V^{*m}(N) \cup V^{*m}(L) = T$.

Remark 2.2. If M is a Max-top module and N and L are submodules of M, then by [26, Lemma 2.1] and since $V^{*m}(N) = Max(M) \cap V^{*}(N)$, we have $V^{*m}(N) \cup V^{*m}(L) = V^{*m}(J^{m}(N)) \cup V^{*m}(J^{m}(L)) = V^{*m}(J^{m}(N) \cap J^{m}(L))$.

Proposition 2.3. The following statements hold.

(a) $G(\tau_T^{*m}) \neq \emptyset$ if and only if T is closed and is not irreducible subset of $Max(M)$.

(b) $G(\tau_T^{*m}) \neq \emptyset$ if and only if $T = V^{*m}(\Im(T))$ and T is not irreducible subset of $Max(M)$.

(c) If $T = V^{*m}(\Im(T))$ and $\Im(T)$ is not a J-radical prime submodule of M, then $G(\tau_T^{*m}) \neq \emptyset$.

Proof. (a) Straightforward.

(b) Suppose that $G(\tau_T^{*m}) \neq \emptyset$. By part (a), it is enough to show that $T = V^{*m}(\Im(T))$. Clearly, $T \subseteq V^{*m}(\Im(T))$. Next, let $V^{*m}(N)$ be any closed subset of $Max(M)$ containing T. Then $m \subseteq N$ for every $m \in T$ so that $\Im(T) \supseteq N$. Hence, for every $Q \in V^{*m}(\Im(T))$, $Q \supseteq \Im(T) \supseteq N$, namely $V^{*m}(\Im(T)) \subseteq V^{*m}(N)$. It follows that $V^{*m}(\Im(T))$ is the smallest closed subset of $Max(M)$ containing T, hence, $V^{*m}(\Im(T)) = T$.

(c) Suppose that $T = V^{*m}(\Im(T))$ and $\Im(T)$ is not a prime submodule of M. We show that T is not irreducible subset of $Max(M)$. Assume that T is irreducible subset of $Max(M)$. Let $IK \subseteq \Im(T)$. One can easily check that $T \subseteq V^{*m}(IK) \subseteq V^{*m}(K) \cup V^{*m}(IM)$. Since T is irreducible, either $T \subseteq V^{*m}(K)$ or $T \subseteq V^{*m}(IM)$. If $T \subseteq V^{*m}(K)$, then $K \subseteq P$, for all $P \in T$, i.e., $K \subseteq \Im(T)$. If $T \subseteq V^{*m}(IM)$, then $IM \subseteq P$, for all $P \in T$, i.e., $IM \subseteq \Im(T)$. Thus $\Im(T)$ is a prime submodule of M. □

Remark 2.4. We have not been able to find an example to show that the converse of part (c) is not true. This motivates the following question.

Question 2.5. Is $\Im(T)$ not a J-radical prime submodule of M, when $G(\tau_T^{*m}) \neq \emptyset$.

Example 2.6. Put $R := \mathbb{Z}$ and $M := \oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}$. Then by [13, Table of examples 3.1], $Max(M) = Spec(M) = \{p_j M\} = \{\oplus_{i \in \mathbb{N}, i \neq j} \mathbb{Z}/p_i\mathbb{Z}\}$ and M is a Max-top module. Let $T := Max(M)$. Now $V^{*m}(\Im(T)) = V^{*m}((\mathbf{0})) = Max(M)$. Hence $G(\tau^{*m}_{Max(M)}) \neq \emptyset$.

Example 2.7. Set $R := \mathbb{Z}$ and $M := \mathbb{Q} \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z})$. Then by [13, Table of examples 3.1], $Max(M) = \{\mathbb{Q} \oplus (\oplus_{i \in \mathbb{N}, i \neq j} \mathbb{Z}/p_i\mathbb{Z})\}$, $Spec(M) = Max(M) \cup \{(\mathbf{0}) \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z})\}$, and M is a Max-top module. Let $T := Max(M)$. Now $V^{*m}(\mathbb{Q} \oplus (\mathbf{0})) = Max(M)$ so that $G(\tau_T^{*m}) \neq \emptyset$.

The following theorem illustrate some graphical parameters.

Theorem 2.8. *The quasi-Zariski topology-graph $G(\tau_T^{*m})$ is connected and $diam(G(\tau_T^{*m})) \leq 3$. Moreover if $G(\tau_T^{*m})$ contains a cycle, then $gr(G(\tau_T^{*m})) \leq 4$.*

Proof. Suppose $N, K \in V(G(\tau_T^{*m}))$ and they are not adjacent. Then $V^{*m}(N) \cup V^{*m}(K) \neq T$, so there exist $L, V \in V(G(\tau_T^{*m}))$ with

$$V^{*m}(J^m(N) \cap J^m(L)) = V^{*m}(J^m(K) \cap J^m(V)) = T.$$

If $L = V$, then $N - L - K$ is a path of length 2. Thus we assume that $L \neq V$. If $V^{*m}(J^m(V) \cap J^m(V)) = T$, then $N - L - V - K$ is a path of length 3. If $V^{*m}(J^m(L) \cap J^m(V)) \neq T$), then $N - J^m(L) \cap J^m(V) - K$ is a path of length of 2 (if $N = J^m(L) \cap J^m(V)$, then

$$V^{*m}(N) \cup V^{*m}(K) = V^{*m}(L) \cup V^{*m}(V) \cup V^{*m}(K)$$

so that

$$T = V^{*m}(J^m(V) \cap J^m(K)) = V^{*m}(J^m(L) \cap J^m(V) \cap J^m(K)).$$

Thus $V^{*m}(J^m(N)) \cap V^{*m}(J^m(K)) = T$, a contradiction. Similarly, we have $K \neq J^m(L) \cap J^m(V))$. Now suppose that $gr(G(\tau_T^{*m})) > 4$. Then without loss of generality, we can assume that $gr(G(\tau_T^{*m})) = 5$. Then $N-L-K-V-W-N$ is a 5-cycle. Clearly, $V^{*m}(L) \cup V^{*m}(V) \neq T$ (resp. $V^{*m}(K) \cup V^{*m}(W) \neq T$). Now one can see that $N - J^m(L) \cap V^{*m}(V) - W - N$ (resp. $N - L - J^m(K) \cap J^m(W) - N$) is a 4-cycle, a contradiction. So we have $gr(G(\tau_T^{*m})) \leq 4$. Hence, the proof is completed. □

Proposition 2.9. Let M be an R-module and $\psi : Max(M) \to Max(R/Ann(M))$ be the natural map. Suppose $Max(M)$ is homeomorphic to $Max(R/Ann(M))$ under ψ. Let $(N : M)M$ and $(L : M)M$ be adjacent in $G(\tau_T^{*m})$ and let $T' = \{\overline{(Q : M)} : Q \in T\}$. Then $\overline{(N : M)}$ and $\overline{(L : M)}$ are adjacent in $G(\tau_{T'}^{*m})$. Conversely, if $\overline{I}$ and $\overline{J}$ are adjacent in $G(\tau_{T'}^{*m})$, then IM and JM are adjacent in $G(\tau_T^{*m})$.

Proof. Since ψ is Max-injective, $\psi^{-1}(T') = T$. Also we have $V^{*m}((N : M)M) \cup V^{*m}((L : M)M) = T$. Hence $\psi(V^{*m}((N : M)M)) \cup \psi(V^{*m}((L : M)M)) = T'$. This implies that $V(\overline{N : M}) \cup V(\overline{L : M}) = T'$ (note that $V^{*m}((N : M)M) = T \iff V(\overline{N : M}) = T'$). Conversely, suppose $V(\overline{I}) \cup V(\overline{J}) = T'$. Then $\psi^{-1}(V(\overline{I})) \cup \psi^{-1}(V(\overline{J})) = T$ so that $V^{*m}(IM) \cup V^{*m}(JM) = T$ (note that $V^{*m}(\overline{I}) = T' \Leftrightarrow V^{*m}(IM) = T$). □

Lemma 2.10. Let $G(\tau_T^{*m}) \neq \emptyset$ and let $Q \in T$. Then Q is a vertex if each of the following statements holds.

(a) There exists a subset T' of T such that $Q \in T'$, $V(\cap_{P\in T'} P) = T$, and $V(\cap_{P\in T', P\neq Q} P) \neq T$. In particular, this holds when T is a finite set.

(b) For a submodule N of M, $N \in V(G(\tau^{*m}_T))$ and $J^m(N) \cap Q \notin V(G(\tau^{*m}_T))$.

Proof. Straightforward. $\square$

Example 2.11. Consider Example 2.7, if $|T| \geq 2$ and $T \subseteq \{\mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}, i\neq 1} \mathbb{Z}/p_i\mathbb{Z})$ $, ..., \mathbb{Q} \oplus (\oplus_{i\in\mathbb{N}, i\neq n} \mathbb{Z}/p_i\mathbb{Z})\}$, then every element of T is a vertex.

Definition 2.12. We define a subgraph $G_d(\tau^{*m}_T)$ of $G(\tau^{*m}_T)$ with vertices $V((G_d(\tau^{*m}_T)))= \{N < M :$ there exists $L < M$ such that $V^{*m}(N) \cup V^{*m}(L) = T$ and $V^{*m}(N), V^{*m}(L) \neq T$ and $V^{*m}(N) \cap V^{*m}(L) = \emptyset\}$, where distinct vertices N and L are adjacent if and only if $V^{*m}(N) \cup V^{*m}(L) = T$ and $V^{*m}(N) \cap V^{*m}(L) = \emptyset$. It is clear that the degree of every $N \in V((G_d(\tau^{*m}_T)))$ is the number of submodules K of M such that $V^{*m}(L) = V^{*m}(K)$, where L is adjacent to N.

We need the following remark.

Remark 2.13. We recall that the *Zariski topology* on $Max(M)$ is the topology τ^m_M described by taking the set $Z^m(M) = \{V(N) : N \leq M\}$ as the set of closed sets of $Max(M)$, where $V(N) = \{P \in Spec(M) : (P : M) \supseteq (N : M)\}$ [23]. If M is a multiplication module then $\tau_M = \tau^{*m}_M$ by [26, Theorem 3.5].

Proposition 2.14. The following statements hold when T is a closed subset of $Max(M)$.

(a) $G_d(\tau^{*m}_T) \neq \emptyset$ if and only if $T = V^{*m}(\Im(T))$ and T is disconnected.

(b) Suppose $\hat{M}$ is an Artinian module and $Spec(\hat{M}) = Max(\hat{M})$. Then $G_d(\tau^{*m}_T) = \emptyset$ if and only if $R/Ann(\hat{M})$ contains no idempotent other than $\bar{0}$ and $\bar{1}$.

Proof. (a) is straightforward.

(b) Since $\hat{M}$ is an Artinian module, then $\hat{M}/rad(\hat{M})$ is a Noetherian module by [18, Corollary 2.30]. As $\hat{M}/rad(\hat{M})$ is a finitely generated top module, it is a multiplication module by [26, Theorem 3.5]. It follows that $\tau_{\hat{M}/rad(\hat{M})} = \tau^{*m}_{\hat{M}/rad(\hat{M})}$ by Remark 2.13. So $\tau_{\hat{M}} = \tau^{*m}_{\hat{M}}$ because $\hat{M}$ and $\hat{M}/rad(\hat{M})$ are homeomorphic by Lemma 4.1. Also the natural map of $\hat{M}/rad(\hat{M})$ is surjective (for, $\hat{M}/rad(\hat{M})$ is finitely generated). Hence the natural map of $\hat{M}$ is surjective by the above arguments. Now the result follows from [23, Corollary 3.8]. $\square$

Theorem 2.15. *$G_d(\tau^{*m}_T)$ is a bipartite graph.*

Proof. At first we assume that $G_d(\tau_T^{*m})$ contains a cycle, we show that $gr(G_d(\tau_T^{*m})) \leq 4$. Without loss of generality, we assume that $gr(G(\tau_T)) = 5$. Then $N - L - K - W - V - N$ is a 5-cycle. It follows that N and W are adjacent so that $N - L - K - W - N$ is a 4-cycle, a contradiction. Now, by [28], G is a bipartite graph if and only if it does not contain an odd cycle. Hence by Theorem **??**, it is enough to show that $G_d(\tau_T) \neq 3$. Suppose $N - L - K - N$ is a 3-cycle. Then $\emptyset = (V^{*m}(N) \cap V^{*m}(L)) \cup (V^{*m}(N) \cap V^{*m}(K)) = V^{*m}(N) \cap (V^{*m}(L) \cup V^{*m}(K)) = V^{*m}(N) \cap T = V^{*m}(N)$. Hence $V(N) = \emptyset$, a contradiction. □

Corollary 2.16. By Theorem 2.15, if $G_d(\tau_T^{*m})$ contains a cycle, then $gr(G_d(\tau_T^{*m})) = 4$.

Example 2.17. Set $R := \mathbb{Z}$ and $M := \mathbb{Z}/12\mathbb{Z}$. So $Spec(M) = Max(M) = \{2\mathbb{Z}/12\mathbb{Z}, 3\mathbb{Z}/12\mathbb{Z}\}$. Set $T := Max(M)$. Clearly, $G(\tau_T^{*m}) = G_d(\tau_T^{*m})$ is a bipartite graph and $\mathbb{Z}/(\cap_{P \in T} P : M) \cong \mathbb{Z}/6\mathbb{Z}$ contains idempotents other than $\overline{0}$ and $\overline{1}$.

Example 2.18. Set $R := \mathbb{Z}$ and $M := \mathbb{Z}/30\mathbb{Z}$. So $Spec(M) = Max(M) = \{2\mathbb{Z}/30\mathbb{Z}, 3\mathbb{Z}/30\mathbb{Z}, 5\mathbb{Z}/30\mathbb{Z}\}$. Set $T := Max(M)$. Clearly, $G_d(\tau_T^{*m})$ is a bipartite graph and $\mathbb{Z}/(\cap_{P \in T} P : M) \cong \mathbb{Z}/30\mathbb{Z}$ contains idempotents other than $\overline{0}$ and $\overline{1}$.

The above example shows that $G_d(\tau_T^{*m})$ is not always connected.

Proposition 2.19. The following statements hold.

(a) $G_d(\tau_T^{*m})$ with two parts U and V is a complete bipartite graph if and only if for every $N, L \in U$ (resp. in V), $V^{*m}(N) = V^{*m}(L)$.

(b) $G_d(\tau_T^{*m})$ is connected if and only if it is a complete bipartite graph.

Proof. Use the fact that if N, L are two vertices, then $d(N, L) = 2$ if and only if $V^{*m}(N) = V^{*m}(L)$. □

We end this section with the following question.

Question 2.20. Let $G(\tau_T^{*m}) \neq \emptyset$, where T be an infinite subset of $Max(M)$. Is $T \cap V(G(\tau_T^{*m})) \neq \emptyset$?

3 The Annihilating-submodule graph

As we mentioned before, $AG(M)$ is a graph with vertices $V(AG(M)) = \{N \leq M : NL = 0 \text{ for some } 0 \neq L < M\}$, where distinct vertices N and L are adjacent if and only if $NL = 0$ (here we recall that the product of N and L is defined by $(N : M)(L : M)M$).

The following results reflect some basic properties of the annihilating-submodule graph of a module.

Proposition A ([5, Proposition 3.2]). Let N be a non-zero proper submodule of M.

(a) N is a vertex in $AG(M)$ if $Ann(N) \neq Ann(M)$ or $(0 :_M (N : M)) \neq 0$.

(b) N is a vertex in $AG(M)$, where M is a multiplication module, if and only if $(0 :_M (N : M)) \neq 0$.

Remark 3.1. In the annihilating-submodule graph $AG(M)$, M itself can be a vertex. In fact M is a vertex if and only if every non-zero submodule is a vertex if and only if there exists a non-zero proper submodule N of M such that $(N : M) = Ann(M)$. For example, when $M = \mathbb{Q}$, then for every submodule N of M (as $\mathbb{Z}$-module), $(N : M) = 0$. Hence M is a vertex in $AG(M)$.

Theorem B ([5, Theorem 3.3]). Assume that M is not a vertex. Then the following hold.

(a) $AG(M)$ is empty if and only if M is a prime module.

(b) A non-zero submodule N of M is a vertex if and only if $(0 :_M (N : M)) \neq 0$.

Theorem C ([5, Theorem 3.4]). The annihilating-submodule graph $AG(M)$ is connected and $diam(AG(M)) \leq 3$. Moreover, if $AG(M)$ contains a cycle, then $gr(AG(M)) \leq 4$.

Proposition D ([5, Proposition 3.4]). The following statements hold.

(a) Let M be a non-simple semisimple R-module. Then every non-zero proper submodule M is a vertex.

(b) Let M be a non-simple homogeneous semisimple R-module. Then $AG(M) = K_\alpha$.

(c) Let M be a prime module with a non-zero socle. Then $AG(M) = \emptyset$ or $AG(M) = K_\alpha$.

(d) Let M be a non-simple module with a non-zero socle. Then $AG(M) \neq \emptyset$. In particular, $AG(M) \neq \emptyset$ when M is a non-simple Artinian module.

Theorem E ([5, Theorem 3.7]). Consider the following statements.

(a) $Ann(M)$ is a prime ideal and M is a divisible $R/Ann(M)$-module.

(b) Every non-zero proper submodule of M is adjacent to M.

(c) For each ideal I of R, we have $IM = M$ or $IM = 0$.

(d) $AG(M) = K_\alpha$.

(e) M is a homogeneous semisimple module.

Then $(a) \longrightarrow (b) \longrightarrow (c) \longrightarrow (d) \longrightarrow (a)$. Moreover, if M is a finitely generated module then $(e) \longleftrightarrow (a)$

4 The relationship between $G(\tau_T^{*m})$ and $AG(M)$

A submodule S of an R-module M will be called semi maximal if S is an intersection of maximal submodules. Further M is called a semi maximal module if $(0) \subseteq M$ is a semi maximal submodule. A proper submodule N of M is said to be semiprime in M, if for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Further M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule. A proper ideal I of R is semiprime if for every ideal J and K of R, $J^2K \subseteq I$ implies that $JK \subseteq I$ [29].

Lemma 4.1. Suppose T is a closed subset of $Max(M)$ equipped with the natural topology induced from of $Max(M)$. Then T and $Max(\hat{M})$ are homeomorphic.

Proof. Let $\phi : Max(\hat{M}) \to T = V^{*m}(\Im(T))$ defined by $\phi(\hat{Q}) = Q$, where $Q \in Max(M)$. Clearly ϕ is a bijection map. We show that ϕ is a continuous map. Let $U = T \cap V^{*m}(N)$ be a closed subset of T, where N is a proper subset of M. Then we have $\phi^{-1}(U) = V^{*m}(\widehat{N + \Im(T)})$. We show that ϕ is closed. Suppose U is a closed subset of $Max(\hat{M})$. Then $U = V^{*m}(\hat{N})$, where $N \leq M$. It is easy to see that $\phi(U) = V^{*m}(N)$. $\square$

One may think that since T and $Max(\hat{M})$ are homeomorphic, the studying $G(\tau_T^{*m})$ can be reduced to studying $G(\tau_{Max(L)}^{*m})$, where L is a semi maximal module. But the following example shows that this is not true.

Example 4.2. Set $R := \mathbb{Z}$, $M := \mathbb{Z}/12\mathbb{Z}$, and $T := Max(M)$. Then $G(\tau_T^{*m}) = K_{1,2}$ but $G(\tau_{Max(M/J^m((0)))}^{*m}) = K_2$.

Remark 4.3. In fact $G(\tau_T^{*m})$ is a non-empty graph if and only if $|E(G(\tau_T^{*m}))| \geq 1$. The following lemma shows that the graph $AG(M)$ has also this property (i.e., $|E(AG(M))| \geq 1$) if M is a semiprime module such that it is not a vertex in $AG(M)$.

Lemma 4.4. Assume that M is not a vertex in $AG(M)$. Then M is a semiprime module if and only if for every non-zero submodule N of M and positive integer number k, $N^k \neq 0$.

Proof. The necessity is clear. To see the converse, let N be a submodule of M and let I be an ideal of R. Let $I^2N = 0$ and $IN \neq 0$. Then we have $(IN)^2 = (IN : M)^2M \subseteq I^2N = 0$, a contradiction. Hence M is a semiprime module. □

Proposition 4.5. The following statements hold.

(a) Suppose N and L are adjacent in $G(\tau_T^{*m})$. Then $\hat{J^m(N)}$ and $\hat{J^m(L)}$ are adjacent in $AG(\hat{M})$.

(b) $G(\tau_T^{*m})$ is isomorphic with a subgraph of $AG(\hat{M})$ or $|E(G(\tau_T^{*m}))| \geq 2$.

Proof. (a) Straightforward.

(b) Assume that $G(\tau_T^{*m})$ is not isomorphic with a subgraph of $AG(\hat{M})$. Hence there exist $N, L \in V(G(\tau_T^{*m}))$ such that N and L are adjacent and $N \neq J^m(N)$. It follows that $N - L - J^m(N)$ is a path of length two. □

Note that an R-module M is fully prime (respectively fully semiprime) if each proper submodule of M is prime (respectively semiprime). In [19, Corollary 1.9], it is shown that M is fully prime (respectively fully semiprime) if and only if is homogeneous semisimple (respectively co-semisimple module).

Theorem 4.6. *The following statements hold.*

(a) Let M be a fully semiprime module. Then $G(\tau_T^{*m})$ is isomorphic with a subgraph of $AG(\hat{M})$.

(b) Let M be a semisimple module and suppose M is not a vertex in $AG(M)$. Then $G(\tau_T^{*m})$ and $AG(\hat{M})$ are isomorphic.

(c) Let M be a homogeneous semisimple module and $Spec(M) = Max(M)$. Then $AG(\hat{M}) = K_\alpha$, where $\alpha = |\Lambda(\hat{M})|$ and $G(\tau_T^{*m}) = \emptyset$.

Proof. (a) By [19, Theorem 2.3], M is a co-semisimple module. So $N = \cap_{P \in V^{*m}(N)} P$, where $N < M$. Hence, by Proposition 4.5 (a), it is easy to see that $G(\tau_T^{*m})$ is isomorphic with a subgraph of $AG(\hat{M})$.

(b) Let M be a semisimple module and suppose M is not a vertex in $AG(M)$. We show that M is a multiplication module. To see this, let N be a proper submodule of M. Then there exists a family $\{T_i, i \in I\}$ of minimal submodules of M such that $N = \oplus_{i \in I} T_i$. Now for each $i \in I$, we have $(T_i : M)M = M$ (note that $(T_i : M)M \neq 0$ because M is not a vertex in $AG(M)$). Hence $N = \oplus_{i \in I}(T_i : M)M = (\oplus_{i \in I}(T_i : M))M$. Thus M is a multiplication module. It follows that if $\hat{N}$ and $\hat{L}$ are adjacent in $AG(\hat{M})$, then N and L are adjacent in $G(\tau_T^{*})$. Since M is a co-semisimple module, by using part (b), we see that $G(\tau_T^{*m})$ is isomorphic with a subgraph of $AG(\hat{M})$. Hence $G(\tau_T^{*m})$ and $AG(\hat{M})$ are isomorphic.

(c) The first assertion follows from Proposition D. To see the second assertion, since $\Im(T)$ is a prime submodule of M (see [19, Corollary 1.9]), we have $G(\tau_T^{*m}) = \emptyset$ by Proposition 2.3 (c).

□

Example 4.7. Put $R := \mathbb{Z}$ and $M := \oplus_{i\in\mathbb{N}}\mathbb{Z}/p_i\mathbb{Z}$. Then by [13, Table of examples 3.1], $Max(M) = Spec(M) = \{p_jM\} = \{\oplus_{i\in\mathbb{N}, i\neq j}\mathbb{Z}/p_i\mathbb{Z}\}$ and M is a top module. $G(\tau^{*m}_{Max(M)})$ is an infinite graph, because every element $\oplus_{i\in\mathbb{N}, i\neq j}\mathbb{Z}/p_i\mathbb{Z}$ of $Max(M)$ is adjacent to $\mathbb{Z}/p_j\mathbb{Z}$. Hence by Theorem 4.6 (b), $AG(M)$ is an infinite graph.

Lemma 4.8. Assume that $\emptyset \neq V(AG(\hat{M}) \subseteq Max(\hat{M})$. Then $|T| = 2$, $AG(\hat{M}) = K_2$ and it is isomorphic with a subgraph of $G(\tau_T^{*m})$.

Proof. Suppose that $\hat{P}$ is a vertex in $AG(\hat{M})$ such that $P \in Max(M)$. Then there exists non-zero proper submodule $\hat{Q}$ of $\hat{M}$ such that it is adjacent to $\hat{P}$, where, $Q \in Max(M)$. One can see that $(P : M) \subseteq (P' : M)$ or $(Q : M) \subseteq (P' : M)$ for every $P' \in T$. Now since $\hat{M}$ is a top module, by [26, Theorem 3.5] $P = P'$ or $Q = P'$. Hence $V^{*m}(P) \cup V^{*m}(Q) = T$. It follows that $|T| = 2$, $AG(\hat{M})$ has only one edge and it is isomorphic with a subgraph of $G(\tau_T^{*m})$. □

Proposition 4.9. Assume that $G(\tau_T^{*m}) \neq \emptyset$.

(a) If $\hat{M}$ is a Noetherian R-module, then $T = V^{*m}(P_1 \cap ... \cap P_n)$.

(b) If $\hat{M}$ is an Artinian top R-module, then $T = V^{*m}(P_1 \cap ... \cap P_n)$, where for each $(1 \leq i \leq n)$, P_i is a vertex. In particular, $|T| = n$.

Proof. (a) Since $\hat{M}$ is a Noetherian module, $\hat{M}$ has a finite number of minimal prime submodules by [27, Theorem 4.2]. Hence $\hat{M} = V^{*m}(\hat{P}_1) \cup ... \cup V^{*m}(\hat{P}_n)$, where each i $(1 \leq i \leq n)$, $\hat{P}_i$ is a minimal prime submodule of $\hat{M}$ and P_i is a prime submodule of M. So by Lemma 4.1, we have $T = V^{*m}(P_1) \cup ... \cup V^{*m}(P_n)$.

(b) As in the proof of Proposition 2.14 (b), $\hat{M}/rad(\hat{M})$ is a Noetherian module. So $\hat{M}/rad(\hat{M})$ has a finite number of minimal prime submodules. Hence $\hat{M}$ has a finite number of minimal prime submodules. So we have $T = V^{*m}(P_1) \cup ... \cup V^{*m}(P_n)$ by part (a). To see the second assertion, we note that since $\hat{M}/rad(\hat{M})$ is a finitely generated top module, it is a multiplication module by [26, Theorem 3.5]. It follows that $\hat{M}/rad(\hat{M})$ is a cyclic Artinian module by [21, Corollary 2.9] and hence $Spec(\hat{M}/rad(\hat{M})) = Max(\hat{M}/rad(\hat{M}))$. So $Spec(\hat{M}) = Max(\hat{M})$. Hence by the above arguments, we have $|T| = n$ and the proof is completed. □

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